Jordan cells in logarithmic limits of conformal field theory

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Abstract

It is discussed how a limiting procedure of conformal field theories may result in logarithmic conformal field theories with Jordan cells of arbitrary rank. This extends our work on rank-two Jordan cells. We also consider the limits of certain three-point functions and find that they are compatible with known results. The general construction is illustrated by logarithmic limits of minimal models in conformal field theory. Characters of quasi-rational representations are found to emerge as the limits of the associated irreducible Virasoro characters.

Keywords: Logarithmic conformal field theory, minimal models, Virasoro characters.

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1 Introduction

Results on logarithmic conformal field theory (CFT) (see [1, 2, 3] for recent reviews of logarithmic CFT, and [4] for a survey on ordinary CFT) are often worked out for Jordan cells of rank two followed by a statement that the results may be extended to higher-rank Jordan cells. This is not always ensured a priori, however. Our construction of rank-two Jordan cells in logarithmic limits of CFTs [5] provides an example where the extension is quite non-trivial. Its resolution is discussed in the present paper. Certain indecomposable representations involving Jordan cells of rank three are analyzed in [6]. Jordan cells of infinite rank have been introduced in [7], while affine Jordan cells relevant to logarithmic extensions of Wess-Zumino-Witten models have been constructed in [8].

The idea of the present construction is to consider a sequence of conformal models labelled by an integer n, with focus on a multiplet of primary fields in each conformal model appearing in the sequence. Each multiplet consists of r fields where r is the rank of the Jordan cell we wish to construct. To get a firmer grip on this, we introduce sequences of primary fields and organize the former in equivalence classes. For finite n, the fields in a given multiplet must all have different conformal weights, while the weights of the associated sequences converge to the same (finite) conformal weight, Δ, as n approaches infinity. A Jordan-cell structure of rank r emerges if one considers a particular linear and (for finite n) invertible map of the multiplet of fields (or of the associated multiplet of sequences) into a new multiplet of r fields. Since the original fields have different conformal weights, the new fields do not all have well-defined conformal weights. In the limit $n \to \infty$, the linear map is singular and thus not invertible (thereby mimicking the Inönü-Wigner or Saletan contractions known from the theory of Lie algebras), while the new multiplet of fields make up a rank-r Jordan cell of conformal weight Δ.

The two-point functions of the new fields are discussed in generality. Particular three-point functions are also considered and found to be compatible with known results on logarithmic CFT.

To further support the idea that a logarithmic CFT may appear as the limit of a sequence of minimal models, we study the limits of the corresponding sequences of irreducible Virasoro characters. We find that the characters of the so-called quasi-rational representations naturally appear as the limiting characters. These quasi-rational representations are believed to play an important role in logarithmic CFT, see [9, 10], for example. Here we discuss certain aspects of the structure of these representations, in particular in regards to singular vectors and irreducibility. We also address how indecomposable representations based on Jordan cells can emerge as a result of the limiting procedure, and discuss the associated characters.

Section 2 concerns the general construction of higher-rank Jordan cells in logarithmic CFT obtained as a limiting procedure of ordinary CFTs. The construction is illustrated by logarithmic limits of minimal models in Section 3 which also contains a discussion of quasi-rational representations as limits of irreducible representations. Concluding remarks may be found in Section 4.

2 Logarithmic limits

A Jordan cell of rank $r = \rho + 1$ consists of one primary field, Ψ_0, and ρ logarithmic partners, $\Psi_1, ..., \Psi_\rho$, satisfying [11]

$$T(z)\Psi_j(w) \;=\; \frac{\Delta\Psi_j(w) + (1-\delta_{j,0})\Psi_{j-1}(w)}{(z-w)^2} + \frac{\partial_w\Psi_j(w)}{z-w}, \qquad j = 0, 1, ..., \rho \tag{1}$$

where $\Psi_{-1} \equiv 0$. Their two-point functions read

$$\begin{aligned}
\langle\Psi_i(z)\Psi_j(w)\rangle &= 0, \qquad i + j < \rho \\
\langle\Psi_i(z)\Psi_\rho(w)\rangle &= \frac{\sum_{m=0}^{i}\frac{(-2)^m}{m!}A_{i-m}\left(\ln(z-w)\right)^m}{(z-w)^{2\Delta}} \\
\langle\Psi_i(z)\Psi_j(w)\rangle &= \langle\Psi_{i+j-\rho}(z)\Psi_\rho(w)\rangle \\
&= \frac{\sum_{m=0}^{i+j-\rho}\frac{(-2)^m}{m!}A_{i+j-\rho-m}\left(\ln(z-w)\right)^m}{(z-w)^{2\Delta}}, \qquad i + j \geq \rho
\end{aligned} \tag{2}$$

Our goal is to construct such a system in the limit of a sequence of ordinary CFTs. We shall work with real structure constants A_j, $j = 0, ..., \rho$, and assume that $A_0 > 0$.

Let us consider a sequence of conformal models M_n, $n \in \mathbb{Z}_>$, with central charges, c_n, converging to the finite value

$$\lim_{n\to\infty} c_n \;=\; c \tag{3}$$

It is assumed that M_n contains a multiplet of r primary fields, $\varphi_{0;n}, ..., \varphi_{\rho;n}$, that is,

$$T_n(z)\varphi_{j;n}(w) \;=\; \frac{\Delta_{j;n}\varphi_{j;n}(w)}{(z-w)^2} + \frac{\partial_w\varphi_{j;n}(w)}{z-w} \tag{4}$$

where T_n is the Virasoro generator in M_n. The conformal weights for given n are all different and may be written

$$\Delta_{j;n} \;=\; \Delta + a_{j;n}, \qquad 0 \leq j \leq \rho \tag{5}$$

with Δ independent of n. The multiplets are organized or ordered so that

$$a_{0;n} > a_{1;n} > ... > a_{\rho;n} \tag{6}$$

Despite these differences, the corresponding sequences of conformal weights all approach Δ, as we require that

$$\lim_{n\to\infty} a_{j;n} \;=\; 0, \qquad 0 \leq j \leq \rho \tag{7}$$

We are thus considering multiplets (labelled by j, $0 \leq j \leq \rho$) of sequences of primary fields such as $(\varphi_{j;1}, \varphi_{j;2}, ...)$, where the element $\varphi_{j;n}$ belongs to M_n. The associated two-point functions are of the form

$$\langle\varphi_{i;n}(z)\varphi_{j;n}(w)\rangle \;=\; \frac{\delta_{ij}C_{j;n}}{(z-w)^{2\Delta_{j;n}}} \tag{8}$$

where, for simplicity, one may normalize the fields so that the non-vanishing structure constants, $C_{j;n}$, are *independent* of n. This is the preferred choice in [5] on rank-two Jordan cells but is not necessary and is *not adopted* here. For simplicity, we shall assume, though, that $C_{j;n} > 0$.

Consider now the linear and invertible map

$$\Psi_{i;n} \;=\; \sum_{j=0}^{\rho} F_{i,j;n}\varphi_{j;n} \tag{9}$$

governed by a sequence of invertible $r \times r$ matrices, F_n. We shall be interested in the limit

$$\Psi_j := \lim_{n\to\infty} \Psi_{j;n} \tag{10}$$

and look for a sequence of maps (9) that would result in a Jordan-cell structure of rank r as n approaches infinity. That is, we must work out

$$\begin{aligned} T(z)\Psi_j(w) &= \lim_{n\to\infty} \{T_n(z)\Psi_{j;n}(w)\} \\ \langle\Psi_i(z)\Psi_j(w)\rangle &= \lim_{n\to\infty} \{\langle\Psi_{i;n}(z)\Psi_{j;n}(w)\rangle\} \end{aligned} \tag{11}$$

and extract conditions on F_n from a comparison with (1) and (2), and ultimately try to find a suitable sequence of such maps.

To this end we shall denote the diagonal matrix $\mathrm{diag}[a_{0;n}, ..., a_{\rho;n}]$ by a_n and introduce the off-diagonal $r \times r$ matrix

$$P \;=\; \begin{pmatrix} 0 & 0 & 0 & ... & 0 & 0 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 & 0 \\ \vdots & & & \vdots & & & \vdots \\ \vdots & & & \vdots & & & \vdots \\ 0 & 0 & 0 & ... & 1 & 0 & 0 \\ 0 & 0 & 0 & ... & 0 & 1 & 0 \end{pmatrix} \tag{12}$$

where $P_{i,j} = \delta_{i,j+1}$. The conditions arising from (1) then read

$$\lim_{n\to\infty} \left\{F_n a_n F_n^{-1}\right\} \;=\; P \tag{13}$$

while the conditions from the two-point functions (2) are

$$\lim_{n\to\infty} \left\{\sum_{\ell=0}^{\rho} C_{\ell;n}F_{i,\ell;n}F_{j,\ell;n}a_{\ell;n}^m\right\} \;=\; \begin{cases} A_{i+j-\rho-m}, & 0 \leq m \leq i+j-\rho \\ \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

The solution for F_n is not unique. The following is the simplest one we have found:

$$\begin{aligned} F_{j,k;n} &= 0, && 0 \le j < k \le \rho \\ F_{k,k;n} &= \sqrt{(-1)^k} \times \sqrt{\alpha_{k;n} \prod_{\ell=1}^{\rho-k} (a_{k;n} - a_{k+\ell;n})}, && 0 \le k \le \rho \\ F_{j,k;n} &= \frac{F_{k,k;n}}{\prod_{\ell=1}^{j-k} (a_{k;n} - a_{k+\ell;n})}, && 0 \le k < j \le \rho \end{aligned} \quad (15)$$

where

$$\alpha_{k;n} = \frac{1}{C_{k;n}} \sum_{j=0}^{k} \frac{(-1)^j A_j}{\prod_{\ell=1}^{k-j} (a_{j+\ell-1;n} - a_{k;n})}, \qquad 0 \le k \le \rho \quad (16)$$

The factor $\sqrt{(-1)^k}$ corresponds to multiplication by i when k is odd or by 1 when k is even. Up to this explicit and potentially imaginary factor, all entries of F_n are non-negative and real for large n. This follows from (6) and that $C_{k;n}$ and A_0 have been assumed positive. Allowing these structure constants to be negative (or even complex) would merely affect the transparency of the notation in (15).

To prove that (15) indeed respects the conditions (13) and (14), we first factorize F_n as

$$F_n = \mathcal{A}_n \times \mathrm{diag}[F_{0,0;n}, F_{1,1;n}, ..., F_{\rho,\rho;n}] \quad (17)$$

where the $r \times r$ matrix $\mathcal{A}_n$ is given by

$$\begin{aligned} \mathcal{A}_{i,j;n} &= 0, && 0 \le i < j \le \rho \\ \mathcal{A}_{j,j;n} &= 1, && 0 \le j \le \rho \\ \mathcal{A}_{i,j;n} &= \frac{1}{\prod_{k=j+1}^{i} (a_{j;n} - a_{k;n})}, && 0 \le j < i \le \rho \end{aligned} \quad (18)$$

Its inverse is found to be

$$\begin{aligned} \mathcal{A}^{-1}_{i,j;n} &= 0, && 0 \le i < j \le \rho \\ \mathcal{A}^{-1}_{j,j;n} &= 1, && 0 \le j \le \rho \\ \mathcal{A}^{-1}_{i,j;n} &= \frac{(-1)^{i-j}}{\prod_{k=j}^{i-1} (a_{k;n} - a_{i;n})}, && 0 \le j < i \le \rho \end{aligned} \quad (19)$$

By inserting (17) into the condition (13), the latter is seen to become independent of the various structure constants as it reduces to

$$\lim_{n\to\infty} \left\{ \mathcal{A}_n a_n \mathcal{A}_n^{-1} \right\} = P \quad (20)$$

The non-trivial part reads

$$\delta_{i,j+1} = \lim_{n\to\infty} \left\{ \frac{\sum_{k=j}^{i} (-1)^{k-j} a_{k;n} \prod_{j\le u<v\le i; u,v\ne k} (a_{u;n} - a_{v;n})}{\prod_{j\le u<v\le i} (a_{u;n} - a_{v;n})} \right\} \quad (21)$$

Likewise, the left side of (14) becomes

$$\lim_{n\to\infty}\left\{\sum_{\ell=0}^{\rho} C_{\ell;n}F_{i,\ell;n}F_{j,\ell;n}a_{\ell;n}^m\right\} = \lim_{n\to\infty}\left\{\sum_{\ell=0}^{\min(i,j)}(-1)^\ell\sum_{t=0}^{\ell}\frac{(-1)^tA_t}{\prod_{s=1}^{\ell-t}(a_{s+t-1;n}-a_{\ell;n})}\right.$$
$$\left.\times\frac{a_{\ell;n}^m\prod_{k=1}^{\rho-\ell}(a_{\ell;n}-a_{\ell+k;n})}{\prod_{u=1}^{i-\ell}(a_{\ell;n}-a_{\ell+u;n})\times\prod_{v=1}^{j-\ell}(a_{\ell;n}-a_{\ell+v;n})}\right\} \tag{22}$$

which is obviously symmetric in i and j (as (14) is). We may thus choose to consider $j\leq i$ in which case the non-trivial part of the condition (14) reduces to

$$\delta_{i+j,\rho+m+t} \tag{23}$$
$$= \lim_{n\to\infty}\left\{\sum_{\ell=t}^{j}(-1)^{\ell-t}\frac{a_{\ell;n}^m\prod_{k=1}^{\rho-i}(a_{\ell;n}-a_{i+k;n})}{\prod_{v=1}^{j-\ell}(a_{\ell;n}-a_{\ell+v;n})\times\prod_{s=1}^{\ell-t}(a_{s+t-1;n}-a_{\ell;n})}\right\}$$
$$= \lim_{n\to\infty}\left\{\frac{\sum_{\ell=t}^{j}(-1)^{\ell-t}a_{\ell;n}^m\prod_{k=1}^{\rho-i}(a_{\ell;n}-a_{i+k;n})\times\prod_{t\leq u<v\leq j;u,v\neq\ell}(a_{u;n}-a_{v;n})}{\prod_{t\leq u<v\leq j}(a_{u;n}-a_{v;n})}\right\} \tag{24}$$

To finally verify (21) and (24), one may employ the following simple observation: Let $Q_N(x_1,...,x_M)$ denote a homogeneous polynomial in the M variables $x_1,...,x_M$. Its degree is bounded by N as $\deg(Q_N)\leq N$, and $Q_N\equiv 0$ for $N<0$. If, for every pair (i,j) with $1\leq i<j\leq M$, $Q_N=0$ when $x_i=x_j$, we may conclude that

$$Q_N(x_1,...,x_M) = \prod_{1\leq i<j\leq M}(x_i-x_j)\times Q_{N-\frac{1}{2}M(M-1)}(x_1,...,x_M) \tag{25}$$

Here $Q_{N-\frac{1}{2}M(M-1)}$ may be zero. This is obviously the case if $N<\frac{1}{2}M(M-1)$. That the Kronecker delta functions in (21) and (24) appear without prefactors, is easily checked.

In conclusion, we have found that Jordan cells of arbitrary rank may emerge in limits of certain sequences of ordinary CFTs.

The solution for rank-two Jordan cells was found in [5]. With the notation used here, it is given by

$$\begin{aligned} F_{0,0;n} &= \sqrt{\frac{A_0(a_{0;n}-a_{1;n})}{C_{0;n}}}, & F_{0,1;n} &= 0 \\ F_{1,0;n} &= \sqrt{\frac{A_0}{C_{0;n}(a_{0;n}-a_{1;n})}}, & F_{1,1;n} &= i\sqrt{\frac{A_0-A_1(a_{0;n}-a_{1;n})}{C_{1;n}(a_{0;n}-a_{1;n})}} \end{aligned} \tag{26}$$

For rank three, the solution reads

$$F_{0,0;n} = \sqrt{\frac{A_0(a_{0;n}-a_{1;n})(a_{0;n}-a_{2;n})}{C_{0;n}}}$$

$$
\begin{aligned}
F_{1,0;n} &= \sqrt{\frac{A_0(a_{0;n}-a_{2;n})}{C_{0;n}(a_{0;n}-a_{1;n})}} \\
F_{2,0;n} &= \sqrt{\frac{A_0}{C_{0;n}(a_{0;n}-a_{1;n})(a_{0;n}-a_{2;n})}} \\
F_{1,1;n} &= i\sqrt{\frac{\{A_0-A_1(a_{0;n}-a_{1;n})\}(a_{1;n}-a_{2;n})}{C_{1;n}(a_{0;n}-a_{1;n})}} \\
F_{2,1;n} &= i\sqrt{\frac{A_0-A_1(a_{0;n}-a_{1;n})}{C_{1;n}(a_{0;n}-a_{1;n})(a_{1;n}-a_{2;n})}} \\
F_{2,2;n} &= \sqrt{\frac{A_0-\{A_1-A_2(a_{1;n}-a_{2;n})\}(a_{0;n}-a_{2;n})}{C_{2;n}(a_{0;n}-a_{2;n})(a_{1;n}-a_{2;n})}} \\
F_{0,1;n} &= F_{0,2;n} \;=\; F_{1,2;n} \;=\; 0
\end{aligned}
\tag{27}
$$

2.1 Three-point functions

The objective here is to indicate that our construction is compatible with known results on three-point functions in logarithmic CFT. We only consider the coupling of three identical logarithmic fields in a Jordan cell of rank two, but find that the results provide further evidence to the sensibility of our construction.

Let Ψ denote the only logarithmic field in the rank-two Jordan cell resulting in the limiting procedure outlined above. The three-point function of our interest thus reads

$$
\begin{aligned}
&\langle\Psi(z_1)\Psi(z_2)\Psi(z_3)\rangle \\
&\quad= \lim_{n\to\infty}\{\langle(F_{1,0;n}\varphi_{0;n}(z_1)+F_{1,1;n}\varphi_{1;n}(z_1))\times(F_{1,0;n}\varphi_{0;n}(z_2)+F_{1,1;n}\varphi_{1;n}(z_2)) \\
&\qquad\times(F_{1,0;n}\varphi_{0;n}(z_3)+F_{1,1;n}\varphi_{1;n}(z_3))\rangle\}
\end{aligned}
\tag{28}
$$

Using that the structure constants of the three-point functions

$$
\begin{aligned}
&\langle\varphi_{j_1;n}(z_1)\varphi_{j_2;n}(z_2)\varphi_{j_3;n}(z_3)\rangle \\
= &\frac{C_{j_1,j_2,j_3;n}}{(z_1-z_2)^{\Delta_{1;n}+\Delta_{2;n}-\Delta_{3;n}}(z_2-z_3)^{-\Delta_{1;n}+\Delta_{2;n}+\Delta_{3;n}}(z_1-z_3)^{\Delta_{1;n}-\Delta_{2;n}+\Delta_{3;n}}}
\end{aligned}
\tag{29}
$$

are symmetric for each given n, and introducing the common abbreviation $z_{ij}=z_i-z_j$, we find

$$
\begin{aligned}
\langle\Psi(z_1)\Psi(z_2)\Psi(z_3)\rangle \;&=\; \frac{1}{(z_{12}z_{23}z_{13})^{\Delta}} \\
\times\;\lim_{n\to\infty}\{&\big[F_{1,0;n}^3C_{0,0,0;n}+3F_{1,0;n}^2F_{1,1;n}C_{1,0,0;n}+3F_{1,0;n}F_{1,1;n}^2C_{1,1,0;n}+F_{1,1;n}^3C_{1,1,1;n}\big] \\
+&\big[-a_{0;n}F_{1,0;n}^3C_{0,0,0;n}-(2a_{0;n}+a_{1;n})F_{1,0;n}^2F_{1,1;n}C_{1,0,0;n}
\end{aligned}
$$

$$
\begin{aligned}
&-(a_{0;n}+2a_{1;n})F_{1,0;n}F^2_{1,1;n}C_{1,1,0;n}-a_{1;n}F^3_{1,1;n}C_{1,1,1;n}\big]\ln(z_{12}z_{23}z_{13})\\
&+\Big[\frac{1}{2}a^2_{0;n}F^3_{1,0;n}C_{0,0,0;n}+(2a^2_{0;n}-2a_{0;n}a_{1;n}+\frac{3}{2}a^2_{1;n})F^2_{1,0;n}F_{1,1;n}C_{1,0,0;n}\\
&(\frac{3}{2}a^2_{0;n}-2a_{0;n}a_{1;n}+2a^2_{1;n})F_{1,0;n}F^2_{1,1;n}C_{1,1,0;n}+\frac{1}{2}a^2_{1;n}F^3_{1,1;n}C_{1,1,1;n}\Big]\\
&\times\left(\ln^2(z_{12})+\ln^2(z_{23})+\ln^2(z_{13})\right)\\
&+\big[a_0^2F^3_{1,0;n}C_{0,0,0;n}+(4a_{0;n}a_{1;n}-a^2_{1;n})F^2_{1,0;n}F_{1,1;n}C_{1,0,0;n}\\
&(-a^2_{0;n}+4a_{0;n}a_{1;n})F_{1,0;n}F^2_{1,1;n}C_{1,1,0;n}+a^2_{1;n}F^3_{1,1;n}C_{1,1,1;n}\big]\\
&\times(\ln(z_{12})\ln(z_{23})+\ln(z_{23})\ln(z_{13})+\ln(z_{13})\ln(z_{12}))+\ldots\}
\end{aligned}
\tag{30}
$$

This should be compared to the known results for three-point functions in logarithmic CFT (see [12] and references therein), that is, $\langle\Psi(z_1)\Psi(z_2)\Psi(z_3)\rangle$ should be of the form

$$
\begin{aligned}
\langle\Psi(z_1)\Psi(z_2)\Psi(z_3)\rangle \;=\;& \frac{1}{(z_{12}z_{23}z_{13})^{\Delta}}\{B_0+B_1\ln(z_{12}z_{23}z_{13})\\
&+B_2\left(\ln^2(z_{12})+\ln^2(z_{23})+\ln^2(z_{13})\right)\\
&+B_3\left(\ln(z_{12})\ln(z_{23})+\ln(z_{23})\ln(z_{13})+\ln(z_{13})\ln(z_{12})\right)\}
\end{aligned}
\tag{31}
$$

To facilitate the comparison, we choose the normalization

$$
C_{0;n} \;=\; C_{1;n} \;=\; \frac{A_0}{a_0-a_1} \tag{32}
$$

in which case (26) gives

$$
\begin{aligned}
F^3_{0,0;n} &= 1\\
F^2_{1,0;n}F_{1,1;n} &= i\left(1-\frac{1}{2}\frac{A_1}{A_0}(a_{0;n}-a_{1;n})-\frac{1}{8}\frac{A_1^2}{A_0^2}(a_{0;n}-a_{1;n})^2+\mathcal{O}((a_{0;n}-a_{1;n})^3)\right)\\
F_{1,0;n}F^2_{1,1;n} &= -\left(1-\frac{A_1}{A_0}(a_{0;n}-a_{1;n})\right)\\
F^3_{1,1;n} &= -i\left(1-\frac{3}{2}\frac{A_1}{A_0}(a_{0;n}-a_{1;n})+\frac{3}{8}\frac{A_1^2}{A_0^2}(a_{0;n}-a_{1;n})^2+\mathcal{O}((a_{0;n}-a_{1;n})^3)\right)
\end{aligned}
\tag{33}
$$

Let us also introduce the following expansions of the three-point structure constants (which of course may depend on n)

$$
\begin{aligned}
C_{0,0,0;n} &= \frac{\alpha_0}{(a_{0;n}-a_{1;n})^2}+\frac{\beta_0}{a_{0;n}-a_{1;n}}+\gamma_0+\mathcal{O}(a_{0;n}-a_{1;n})\\
C_{1,0,0;n} &= \frac{\alpha_1}{(a_{0;n}-a_{1;n})^2}+\frac{\beta_1}{a_{0;n}-a_{1;n}}+\gamma_1+\mathcal{O}(a_{0;n}-a_{1;n})\\
C_{1,1,0;n} &= \frac{\alpha_2}{(a_{0;n}-a_{1;n})^2}+\frac{\beta_2}{a_{0;n}-a_{1;n}}+\gamma_2+\mathcal{O}(a_{0;n}-a_{1;n})\\
C_{1,1,1;n} &= \frac{\alpha_3}{(a_{0;n}-a_{1;n})^2}+\frac{\beta_3}{a_{0;n}-a_{1;n}}+\gamma_3+\mathcal{O}(a_{0;n}-a_{1;n})
\end{aligned}
\tag{34}
$$

Based on these, the comparison of (30) with (31) yields

$$B_3 \;=\; -2B_2 \tag{35}$$

and

$$\begin{aligned} -\alpha_0 - i\alpha_1 \;=\; i\alpha_1 - \alpha_2 \;=\; \alpha_2 + i\alpha_3 \;&=\; B_2 \\ \beta_0 + 2i\beta_1 - \beta_2 \;=\; -i\beta_1 + 2\beta_2 + i\beta_3 \;&=\; -B_1 + \frac{A_1}{A_0}B_2 \\ \gamma_0 + 3i\gamma_1 - 3\gamma_2 - i\gamma_3 \;&=\; B_0 + \frac{3}{2}\frac{A_1}{A_0}B_1 - \frac{3}{4}\frac{A_1^2}{A_0^2}B_2 \end{aligned} \tag{36}$$

As non-trivial results, we thus have that B_3 and B_2 are related as in (35), and that the easily solved linear system (36) consists of three decoupled systems. Imposing the consistency condition (35) on (31) corresponds to restricting to the special three-point functions discussed in [13] (see also [14]), cf. [12]. We conclude that our construction appears to yield sensible results for three-point functions.

3 Minimal models

Here we illustrate the general construction above by considering limits of minimal models. The minimal model $\mathcal{M}(p,p')$ is characterized by the coprime integers p and p' which may be chosen to satisfy $p > p' > 1$. The central charge is given by

$$c \;=\; 1 - 6\frac{(p-p')^2}{pp'} \tag{37}$$

whereas the primary fields, $\phi_{r,s}$, have conformal weights given by

$$\Delta_{r,s} \;=\; \frac{(rp-sp')^2-(p-p')^2}{4pp'}, \qquad 1 \leq r < p', \quad 1 \leq s < p \tag{38}$$

The bounds on r and s define the Kac table of admissible primary fields. With the identification

$$\phi_{r,s} \;=\; \phi_{p'-r,p-s} \tag{39}$$

there are $(p-1)(p'-1)/2$ distinct primary fields in the model. These models are unitary provided $p = p'+1$.

We now follow our recent work on rank-two Jordan cells [5]. For each positive integer k, we thus consider the sequence of minimal models $\mathcal{M}(kn+1,n)$, $n \geq 2$. The central charges and conformal weights are given by

$$c^{(k;n)} \;=\; 1 - 6\frac{((k-1)n+1)^2}{n(kn+1)}$$

$$
\begin{aligned}
&= 1-6\frac{(k-1)^2}{k}-6\frac{(k^2-1)}{k^2n}+\mathcal{O}(1/n^2)\\
\Delta_{r,s}^{(k;n)} &= \frac{((kn+1)r-ns)^2-((k-1)n+1)^2}{4n(kn+1)}\\
&= \frac{(kr-s)^2-(k-1)^2}{4k}+\frac{k^2(r^2-1)-(s^2-1)}{4k^2n}+\mathcal{O}(1/n^2) \qquad (40)
\end{aligned}
$$

with limits

$$
\begin{aligned}
c^{(k)} &= \lim_{n\to\infty} c^{(k;n)} = 1-6\frac{(k-1)^2}{k}\\
\Delta_{r,s}^{(k)} &= \lim_{n\to\infty} \Delta_{r,s}^{(k;n)} = \frac{(kr-s)^2-(k-1)^2}{4k}, \qquad r,s\in\mathbb{Z}_> \qquad (41)
\end{aligned}
$$

As discussed in [5], these are seen to correspond to the similar values in the (non-minimal) model $\mathcal{M}(k,1)$ with *extended* Kac table in which r and s are unbounded from above. In particular, the spectrum of the extended model $\mathcal{M}(1,1)$ with central charge $c^{(1)}=1$ is thereby related to the limit of the sequence of *unitary* minimal models $\mathcal{M}(n+1,n)$. A different approach to rank-two Jordan cells and logarithmic CFT based on minimal models may be found in [15].

There is a natural embedding of the Kac table associated to $\mathcal{M}(kn_1+1,n_1)$ into the Kac table associated to $\mathcal{M}(kn_2+1,n_2)$ if $n_1\leq n_2$, mapping $\phi_{r,s}^{(k,n_1)}$ to $\phi_{r,s}^{(k,n_2)}$. It is noted, however, that the conformal weights and representations in general will be altered. Our point here is that if (r,s) is admissible for n_0, it will be admissible for all $n\geq n_0$. We thus have a natural notion of sequences of primary fields: $(\phi_{r,s}^{(k,n_0)},\phi_{r,s}^{(k,n_0+1)},\ldots)$. The parameter n_0 is essentially immaterial since we are concerned with the properties of the sequences as $n\to\infty$. We therefore denote such a sequence simply as $\Upsilon_{r,s}^{(k)}$.

These sequences may be organized in equivalence classes, where $\Upsilon_{r,s}^{(k)}$ and $\Upsilon_{u,v}^{(k)}$ are said to be equivalent if they approach the same conformal weight. According to [5], this presupposes that

$$
I:\quad (u,v) = (r+q,s+kq), \qquad r,s,u,v\in\mathbb{Z}_>,\quad q\in\mathbb{Z} \qquad (42)
$$

or

$$
II:\quad (u,v) = (-r+q,-s+kq), \qquad r,s,u,v\in\mathbb{Z}_>,\quad q\in\mathbb{Z} \qquad (43)
$$

In either case, the approached conformal weight is $\Delta_{r,s}^{(k)}$ given in (41). The equivalence becomes trivial (i.e., $\Upsilon_{r,s}^{(k)}=\Upsilon_{u,v}^{(k)}$) if $q=0$ in case I or if $q=2r$ and $s=kr$ in case II. With the notation (cf. (5))

$$
\Delta_{r,s}^{(k;n)} = \Delta_{r,s}^{(k)}+a_{r,s;n}^{(k)} \qquad (44)
$$

we have

$$
a_{r,s;n}^{(k)}-a_{r+q,s+qk;n}^{(k)} = \frac{q(2n(s-rk)-2r-q)}{4n(kn+1)}
$$

$$
\begin{aligned}
&= \frac{q(s-rk)}{2kn} - \frac{q(2s+qk)}{4k^2n^2} + \mathcal{O}(1/n^3) \\
a^{(k)}_{r,s;n} - a^{(k)}_{-r+q,-s+qk;n} &= \frac{q(2n(rk-s)+2r-q)}{4n(kn+1)} \\
&= \frac{q(rk-s)}{2kn} + \frac{q(2s-qk)}{4k^2n^2} + \mathcal{O}(1/n^3) \qquad (45)
\end{aligned}
$$

According to the general prescription outlined in the previous section, one can now construct a Jordan cell of *arbitrary rank* for each conformal dimension $\Delta^{(k)}_{r,s}$ in the spectrum of the extended $\mathcal{M}(k,1)$ (41), i.e., for each pair (r,s) with $r,s \geq 1$. To avoid confusion with the conventional labelling of primary fields employed here, we shall denote the rank by $\rho+1$. For $rk < s$ one may consider the following multiplet of sequences

$$
\Upsilon^{(k)}_{r+q_0,s+q_0k},\ \Upsilon^{(k)}_{r+q_1,s+q_1k},\ \ldots,\ \Upsilon^{(k)}_{r+q_\rho,s+q_\rho k}, \qquad 0 \leq q_0 < q_1 < \ldots < q_\rho \qquad (46)
$$

It is thus ordered to comply with (6). For $rk \geq s$ one may consider the ordered multiplet

$$
\Upsilon^{(k)}_{r+q_0,s+q_0k},\ \Upsilon^{(k)}_{r+q_1,s+q_1k},\ \ldots,\ \Upsilon^{(k)}_{r+q_\rho,s+q_\rho k}, \qquad q_0 > q_1 > \ldots > q_\rho \geq 0 \qquad (47)
$$

These multiplets are obviously not unique as one may choose to work with case II (43) instead or even combinations of the two cases. This general construction of Jordan cells of rank $\rho+1$ applies to all positive integer k.

3.1 Characters and quasi-rational representations

It is recalled (see [4], for example) that the Virasoro character of the irreducible representation corresponding to (r,s) in the Kac table associated to $\mathcal{M}(p,p')$ may be written

$$
\chi_{r,s}(q) \;=\; K_{pr-p's}(q) - K_{pr+p's}(q) \qquad (48)
$$

where

$$
K_\lambda(q) \;=\; \frac{1}{\eta(q)} \sum_{m\in\mathbb{Z}} q^{(\lambda+2mpp')^2/4pp'} \qquad (49)
$$

with the Dedekind η function given by

$$
\eta(q) \;=\; q^{1/24} \prod_{m=1}^{\infty} (1-q^m) \qquad (50)
$$

For given (r,s), we introduce the sequence of such characters where a single character is defined in $\mathcal{M}(kn+1,n)$ for each $n \geq n_0$. As in the case of sequences of primary fields, n_0 is essentially immaterial since we are concerned with the behaviour as $n \to \infty$. To study this, we examine

$$
\begin{aligned}
K_{(kn+1)r-ns}(q) &= \frac{1}{\eta(q)} \sum_{m\in\mathbb{Z}} q^{((kn+1)r-ns+2mn(kn+1))^2/4n(kn+1)} \\
&= \frac{1}{\eta(q)} \sum_{m\in\mathbb{Z}} q^{\frac{((kn+1)r-ns)^2}{4n(kn+1)}+m((kn+1)r-ns+mn(kn+1))} \qquad (51)
\end{aligned}
$$

and it follows that, as $n \to \infty$, the only finite power of q occurs for $m = 0$. The sequence of characters $\chi_{r,s}(q)$ in $\mathcal{M}(kn+1, n)$ thus approaches

$$\begin{aligned}\chi_{r,s}(q) &\to \frac{1}{\eta(q)} \lim_{n\to\infty} \left(q^{\frac{((kn+1)r-ns)^2}{4n(kn+1)}} - q^{\frac{((kn+1)r+ns)^2}{4n(kn+1)}} \right) \\ &= \frac{1}{\eta(q)} \left(q^{\frac{(kr-s)^2}{4k}} - q^{\frac{(kr+s)^2}{4k}} \right) \end{aligned} \tag{52}$$

This is recognized as the character of the so-called quasi-rational representation $Q_{r,s}^{(k,1)}$ (see (56) and (57) below)

$$\chi_{r,s}^{(k)}(q) = \frac{q^{(1-c^{(k)})/24}}{\eta(q)} q^{\Delta_{r,s}^{(k)}} (1 - q^{rs}) \tag{53}$$

associated to $\mathcal{M}(k, 1)$. As already indicated, these representations and their characters exist for *all* positive pairs (r, s) corresponding to an *extended* Kac table as opposed to the *ordinary* Kac table, cf. (38).

These considerations can be extended straightforwardly to other sequences than our main example $\mathcal{M}(kn+1, n)$. As pointed out to us by A. Nichols and discussed in [5], we could consider a sequence of the form $\mathcal{M}(pp'n+1, p'^2n)$. In this case, we have

$$\begin{aligned} c^{(p,p')} &= \lim_{n\to\infty} c^{(p,p';n)} = \lim_{n\to\infty} \left(1 - 6\frac{(pp'n+1-p'^2n)^2}{(pp'n+1)p'^2n} \right) \\ &= 1 - 6\frac{(p-p')^2}{pp'} \\ \Delta_{r,s}^{(p,p')} &= \lim_{n\to\infty} \Delta_{r,s}^{(p,p';n)} = \lim_{n\to\infty} \frac{(r(pp'n+1) - sp'^2n)^2 - (pp'n+1-p'^2n)^2}{4(pp'n+1)p'^2n} \\ &= \frac{(rp-sp')^2 - (p-p')^2}{4pp'} \end{aligned} \tag{54}$$

These limits correspond to the central charge and the extended Kac table of the model $\mathcal{M}(p, p')$. The limits of the associated sequences of characters are found to be

$$\begin{aligned} \chi_{r,s}^{(p,p')}(q) &= \lim_{n\to\infty} \chi_{r,s}^{(p,p';n)}(q) = \lim_{n\to\infty} \left(K_{(pp'n+1)r-p'^2ns}(q) - K_{(pp'n+1)r+p'^2ns}(q) \right) \\ &= \frac{q^{(1-c^{(p,p')})/24}}{\eta(q)} q^{\Delta_{r,s}^{(p,p')}} (1 - q^{rs}) \end{aligned} \tag{55}$$

corresponding to the quasi-rational representations $Q_{r,s}^{(p,p')}$ labelled by the extended Kac table of $\mathcal{M}(p, p')$.

For every pair of coprime positive integers (p, p') and every pair of positive integers (r, s), the highest-weight Verma module $V_{r,s}^{(p,p')}$ exists. It contains a singular vector at

level rs from which the submodule denoted $V_{r,-s}^{(p,p')}$ is generated. This notation is justified by the simple relation $\Delta_{r,-s}^{(p,p')} = \Delta_{r,s}^{(p,p')} + rs$. The quotient module

$$Q_{r,s}^{(p,p')} \;=\; V_{r,s}^{(p,p')}/V_{r,-s}^{(p,p')} \tag{56}$$

is typically not irreducible but reducible, see below. Due to its properties in regards to fusion [18, 9, 10], it is often referred to as the *quasi-rational* representation $Q_{r,s}^{(p,p')}$. As already announced, its character $\chi(Q_{r,s}^{(p,p')})$ is given by the expression in (55) since

$$\chi(Q_{r,s}^{(p,p')}) \;=\; \frac{q^{(1-c^{(p,p')})/24}}{\eta(q)}\Big(q^{\Delta_{r,s}^{(p,p')}} - q^{\Delta_{r,-s}^{(p,p')}}\Big) \;=\; \frac{q^{(1-c^{(p,p')})/24}}{\eta(q)}q^{\Delta_{r,s}^{(p,p')}}\,(1-q^{rs}) \tag{57}$$

Let us comment on the fate of singular vectors and the emergence of these quasi-rational representations $Q_{r,s}^{(p,p')}$. In the minimal model $\mathcal{M}(\hat{p},\hat{p}')$, the highest-weight Verma module $V_{r,s}^{(\hat{p},\hat{p}')}$ with highest weight $\Delta_{r,s}^{(\hat{p},\hat{p}')}$ is reducible as it contains two distinct singular vectors at levels rs and $(\hat{p}'-r)(\hat{p}-s)$, respectively. The quotient module obtained by dividing out the proper submodules generated from these two vectors is the associated *irreducible* highest-weight module with highest weight $\Delta_{r,s}^{(\hat{p},\hat{p}')}$. Now, in the sequences under consideration, the level $(\hat{p}'-r)(\hat{p}-s)$ of the second singular vector increases as a polynomial function of n (since $(\hat{p},\hat{p}') = (pp'n+1, p'^2n)$, for example). In the limit $n\to\infty$, the vector in question thus appears with infinite weight and is discarded along with all other vectors with infinite weight. The surviving singular vector appears at unchanged level rs. The quotient module obtained by dividing out the submodule generated from this singular vector is the aforementioned quasi-rational quotient module or representation $Q_{r,s}^{(p,p')}$ (which may nevertheless abstain from being irreducible, see below). The limit of irreducible representations with Kac labels (r,s) in the sequence of minimal models $\mathcal{M}(pp'n+1, p'^2n)$, for example, thus corresponds to the quasi-rational representation $Q_{r,s}^{(p,p')}$ with extended Kac labels (r,s) associated to $\mathcal{M}(p,p')$.

This analysis indicates that quasi-rational representations are natural objects in the model constructed as the limit of certain sequences of minimal models. This is in accordance with known results on logarithmic CFT where these quasi-rational representations seem to play an important role, see [9, 10], for example.

To illustrate how these considerations of limits of characters are related to the construction of Jordan cells above, we focus on the sequence $\mathcal{M}(2n+1,n)$, that is, $\mathcal{M}(kn+1,n)$ with $k=2$. The spectrum of the resulting model is thus described in terms of the extended Kac table of $\mathcal{M}(2,1)$. Let us consider the two equivalent sequences $\Upsilon_{1,1}^{(2)}$ and $\Upsilon_{1,3}^{(2)}$, cf. (43). Individually, they give rise to the quasi-rational characters

$$\begin{aligned}
\chi_{1,1}^{(2,1)}(q) &= \frac{q^{1/8}}{\eta(q)}(1-q) \;=\; q^{1/12}(1+q^2+q^3+2q^4+2q^5+\ldots) \\
\chi_{1,3}^{(2,1)}(q) &= \frac{q^{1/8}}{\eta(q)}(1-q^3) \;=\; q^{1/12}(1+q+2q^2+2q^3+4q^4+5q^5+\ldots)
\end{aligned} \tag{58}$$

If combined to form a rank-two Jordan cell, that is, $\varphi_{0;n} = \phi_{1,1}^{(2n+1,n)}$ and $\varphi_{1;n} = \phi_{1,3}^{(2n+1,n)}$, on the other hand, the result is quite different. We first note that $a_{0;n} = a_{1,1;n}^{(2)} = 0$ and $a_{1;n} = a_{1,3;n}^{(2)} = -1/2n + \mathcal{O}(1/n^2)$ thus satisfying (6). For finite n, the energy levels of excitations in $\varphi_{0;n}$ will in general not differ by integers from the energy levels in $\varphi_{1;n}$ since these differences are $|\Delta_{0;n} - \Delta_{1;n}|$ mod an integer. As $n \to \infty$, however, the off-integer parts of these differences vanish. Also for finite n, the action of the Virasoro modes on the two representations is diagonal in the sense that it does not mix the states in the two modules. It is built in by construction, though, that in the limit $n \to \infty$, the action of the Virasoro modes is *non-diagonal*, cf. (1). This means that the resulting module is *indecomposable*. One may view it as an indecomposable combination of the two quasi-rational representations $Q_{1,1}^{(2,1)}$ and $Q_{1,3}^{(2,1)}$ where the diagonal parts of the Virasoro action yield these modules separately while the indecomposable structure is governed by the off-diagonal part.

Since the character of this indecomposable module is insensitive or blind to the off-diagonal structure of the Jordan cell, it is given by the *sum* of the two characters appearing in (58). Denoting this character by $\chi_{(1,1),(1,3)}^{(2,1)}(q)$, we see that is is given by

$$\chi_{(1,1),(1,3)}^{(2,1)}(q) \;=\; \frac{q^{1/8}}{\eta(q)}(2-q-q^3) \;=\; q^{1/12}(2+q+3q^2+3q^3+6q^4+7q^5+\ldots) \tag{59}$$

Possibly up to the finer details of the indecomposable structure, this indecomposable representation has already appeared in the literature. In [9], it is denoted $\mathcal{R}_{1,1}$ while it is denoted $(1,1)\oplus_i(1,3)$ in [10]. Most of the comments above about its emergence as the limit of a combination of two irreducible representations apply generally to our limiting procedure.

Let us also comment on the fact that quasi-rational representations may not be irreducible, even though they can arise as limits of irreducible representations. To appreciate this, we initially consider the Verma module $V_{1,3}^{(4,3)}$ in the minimal model $\mathcal{M}(4,3)$. It has two proper submodules starting at levels $rs = 3$ and $(p'-r)(p-s) = 2$, respectively. The singular vector from which the latter submodule is generated is

$$|\eta_{1,3}^{(4,3)}\rangle \;=\; \Big(L_{-2} - \frac{3}{2(2\Delta_{1,3}^{(4,3)}+1)}L_{-1}^2\Big)|\Delta_{1,3}^{(4,3)}\rangle \tag{60}$$

The quasi-rational module $Q_{1,3}^{(4,3)}$ is not irreducible due to this singular vector. Now, in the sequence $\mathcal{M}(12n+1,9n)$, we are considering $V_{1,3}^{(12n+1,9n)}$ with proper submodules starting at levels 3 and $(9n-1)(12n-2)$, respectively. This means, in particular, that there is no singular vector at level 2. Vectors in $V_{1,3}^{(12n+1,9n)}$ somehow corresponding to $|\eta_{1,3}^{(4,3)}\rangle$ are

$$|\eta_{1,3}^{(12n+1,9n)}\rangle \;=\; \Big(L_{-2} - \frac{3}{2(2\Delta_{1,3}^{(4,3)}+1)}L_{-1}^2\Big)|\Delta_{1,3}^{(12n+1,9n)}\rangle$$

$$|\eta_{1,3}^{(12n+1,9n)}\rangle_{alt} \;=\; \Big(L_{-2}-\frac{3}{2(2\Delta_{1,3}^{(12n+1,9n)}+1)}L_{-1}^2\Big)|\Delta_{1,3}^{(12n+1,9n)}\rangle \qquad (61)$$

though, as already stated, these are not singular. In the limit $n\to\infty$, on the other hand, both of them approach $|\eta_{1,3}^{(4,3)}\rangle$ in $V_{1,3}^{(4,3)}$ thereby rendering the module arising in the limit $n\to\infty$, namely $Q_{1,3}^{(4,3)}$, *reducible*.

4 Conclusion

We have discussed how Jordan cells of arbitrary rank may be constructed in a limiting procedure of ordinary CFTs. Our construction is quite general and has been illustrated by (an infinite family of) sequences of minimal models. It may also be extended to $N=1$ superconformal field theory. This is demonstrated explicitly in the rank-two case in our recent work [5]. A somewhat related construction of rank-two Jordan cells in CFT based on (graded) parafermions is discussed in [16, 17].

It is emphasized that the present work is focused on the mere construction of Jordan cells and the emergence and structure of quasi-rational representations. Questions regarding the resulting models being well-defined logarithmic CFTs will be addressed elsewhere. It is stressed in this context that the only three-point function we have analyzed is (28) for rank two and that the only thing we have verified is that, to the degree of our analysis, it is *compatible* with known results in logarithmic CFT. Nevertheless, this is already a non-trivial test of our construction. A more thorough examination of three-point functions is clearly desirable, in particular in the case of minimal models, though beyond the scope of the present work. It is also stressed that we have only worked with the particular solution (26). A further study may reveal that a more general solution is required. Even so, we find that the current analysis adds substantial new evidence to the suggestion [5] that a logarithmic CFT may result as the limit of a sequence of ordinary CFTs.

It is also emphasized that we are *not* claiming that a logarithmic CFT *must* emerge as the limit of a sequence of CFTs. Instead, we are arguing that a logarithmic CFT *may* appear. As far as the characters go, on the other hand, we have found that characters of quasi-rational representations *always* emerge as the limit of *certain* sequences of irreducible Virasoro characters. It is obvious, though, that not all sequences result in characters of quasi-rational representations.

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